UNIQUENESS OF PAIRINGS IN HOPF-CYCLIC COHOMOLOGY

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ABSTRACT. We show that all pairings defined in the literature extending Connes-Moscovici characteristic map in Hopf-cyclic cohomology are isomorphic as natural transformations of derived double functors.

1. INTRODUCTION

The category of algebras over a fixed ground ring is not an abelian category. This means we are denied the use of the amenities provided by the classical homological algebra on the category of algebras and their morphisms directly. One way around this problem is to find a "good" (faithful) functor from the category of algebras into an abelian category and apply the tools of homological algebra on the image of this functor. Hochschild and cyclic (co)homology are examples of this type where we use Δ-**Mod** the abelian category of simplicial modules for the former and Λ-**Mod** the abelian category of cyclic modules for the latter.

One recurring problem in homological algebra is the task of showing certain functors $F_*, G_*\colon \mathbf{D}(\mathcal{A}) \to$ k-**Mod** on the derived category $\mathbf{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ are isomorphic. This task can be accomplished by finding isomorphic functors F and G which are defined on the underlying abelian category $\mathcal{A}$, and showing that F_* and G_* are obtained as *derivatives* of these functors. If our functors F and G are defined only on a non-abelian subcategory, this approach will not work. The example we have in mind is the category of (co)cyclic modules and the subcategory of (co)cyclic modules coming from the image of the functor $Z \mapsto Cyc_\bullet(Z)$ sending an (co)algebra to its canonical (co)cyclic module.

In the context of cyclic (co)homology of (co)algebras there are various derived categories used in the literature. We note (i) the derived category of cyclic modules [1], (ii) the derived category of mixed complexes [14] and (iii) the homotopy category of towers of super complexes [8] where the last two of are homotopy equivalent by [21]. In this paper, we add another derived category to this list: $\mathbf{D}((\Lambda, \mathbf{T})$-**Mod**$)$ the relative derived category of cyclic modules to implement Connes' very first definition of cyclic cohomology [2, Sect. I, Def. 2] as a derived functor in Theorem 5.12. This derived category will allow us to form a bridge between isomorphic pairings defined in (i) and (ii) [15], and other pairings defined in the literature.

Since Hopf algebras play the role of symmetries of noncommutative spaces, and Hopf-cyclic cohomology extends group and Lie algebra cohomology [4, 5, 6, 11], one should expect existence of cup products in Hopf-cyclic cohomology. There are numerous such products and pairings in the literature [10, 7, 18, 15, 23, 22] which extend the characteristic map defined by Connes and Moscovici [4]. Let H be a Hopf algebra, A be a H-module algebra and M be an arbitrary H-module/comodule. In this paper we prove that the pairings and cup products we enumerated above which extended Connes-Moscovici

characteristic map

$$HC^p_{\mathrm{Hopf}}(H,M) \otimes HC^0_{\mathrm{Hopf}}(A,M) \to HC^p(A)$$

are isomorphic natural transformations of isomorphic double functors defined in their ambient derived categories. Since our setup uses module (co)algebras over a fixed base Hopf-algebra, in addition to the canonical functor $Cyc_\bullet$ which associates an ordinary (co)algebra a (co)cyclic module, we employ another functor $C_\bullet$ [16, Def. 4.7] which is defined from the category of (co)module (co)algebras in to the category of (co)cyclic modules. Because the category of algebras, and therefore the subcategory $im(C_\bullet)$ of the category of (co)cyclic modules, is not abelian we will achieve our objective by finding isomorphic functors on the full double category Λ-**Mod** $\times$ Λ-**Mod** whose derivatives on $\mathbf{D}(\Lambda$-**Mod**$) \times \mathbf{D}(\Lambda$-**Mod**$)$ restricted to $\mathbf{D}(im(C_\bullet)) \times \mathbf{D}(im(C_\bullet))$ yield the pairings we are interested. Thus we reduce the task of showing these pairings are isomorphic in various derived categories, to showing that they come from isomorphic double functors on the abelian category of (co)cyclic modules.

Here is a plan of our paper. In Section 2, we will recall few relevant facts about relative (co)homology à la Hochschild [13]. In Section 3 we develop the necessary machinery for products of abelian categories and their derived categories. We use this machinery in Section 4 to develop a universal pairing using the double functor $diag_\bullet \mathrm{Hom}_k(\,\cdot\,,\,\cdot\,)$ we used extensively in [15] for Connes' cyclic category Λ, this time for modules over an arbitrary small category $\mathbf{C}$. By allowing the base category to change, one can get similar pairing and products for other (co)homology theories. In Section 5 we prove that the relative derived category $\mathbf{D}((\Lambda,\mathbf{T})$-**Mod**$)$ implements cyclic (co)homology via cyclic invariant Hochschild cochains. We also construct a comparison functor $\mathbf{D}(\Lambda$-**Mod**$) \to \mathbf{D}((\Lambda,\mathbf{T})$-**Mod**$)$ between the derived category of (co)cyclic modules and the relative derived category of (co)cyclic modules, which is a homotopy equivalence for a fixed ground field k of characteristic 0. Finally, in Section 6 we prove our uniqueness result as we outlined above.

In this paper we fix a ground field k. We will assume that $char(k) = 0$. All unadorned tensor products $\otimes$ are taken over k. All categories we will use are assumed to be small. All abelian categories are assumed to have enough injectives and projectives.

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2. Relative (Co)homology

In this section we assume R is a unital associative k-algebra. Most of this material can be found in [13].

Definition 2.1. Let S be a subalgebra of R. A morphism of R-modules $f\colon X \to Y$ is called an (R,S)-epimorphism (resp. (R,S)-monomorphism) if (i) f is an epimorphism (resp. monomorphism) of R-modules and (ii) f is a split epimorphism (resp. monomorphism) of S-modules. A short exact sequence $0 \to X \xrightarrow{v} Y \xrightarrow{u} Z \to 0$ of R-modules is called (R,S)-exact if u is an (R,S)-epimorphism and v is an (R,S)-monomorphism.

Definition 2.2. An R-module P is called an (R,S)-projective module if for any (R,S)-epimorphism $u\colon X \to Y$ and morphism of R-modules $p\colon P \to Y$ one can find $\tilde{p}\colon P \to X$ such that $\tilde{p} \circ u = p$.

$$\begin{tikzcd} & P \arrow[d, "p"] \arrow[dl, dotted, "\tilde{p}"'] & \\ X \arrow[r, "u"'] & Y \arrow[r] & 0 \end{tikzcd}$$

Definition 2.3. A k-algebra S is called semi-simple if every monomorphism of S-modules, or equivalently every epimorphism of S-modules, splits.

This definition immediately implies the following

Lemma 2.4. *Let S be a semi-simple subalgebra of R. Then the class of (R,S)-epimorphisms and (R,S)-monomorphisms coincide with the class of ordinary epimorphisms and ordinary monomorphisms of R-modules, respectively.*

Proposition 2.5. *Assume S is a semi-simple subalgebra of an algebra R. Then ordinary Tor groups $\mathrm{Tor}_*^R(X,Y)$ and relative Tor groups $\mathrm{Tor}_*^{(R,S)}(X,Y)$ are naturally isomorphic for an arbitrary right R-module X and an arbitrary left R-module Y. Similarly, ordinary Ext groups $\mathrm{Ext}_R^*(X,Z)$ and the relative Ext groups $\mathrm{Ext}_{(R,S)}^*(X,Z)$ are naturally isomorphic for an arbitrary pair of R-modules (X,Z) of the same parity.*

Proof. Let X be a right R-module and let Y be a left R-module. Since k is a field, the homology of the two sided bar complex $\mathrm{CB}_*(X,R,Y)$ which is $\bigoplus_{n\geq 0} X \otimes R^{\otimes n} \otimes Y$ with the differentials

$$\begin{aligned} d_n^{\mathrm{CB}}(x \otimes r_1 \otimes \cdots \otimes r_n \otimes y) =& (xr_1 \otimes r_2 \otimes \cdots \otimes r_n \otimes y) \\ &+ \sum_{j=1}^{n-1} (-1)^j (x \otimes \cdots \otimes r_j r_{j+1} \otimes \cdots \otimes y) \\ &+ (-1)^n (x \otimes r_1 \otimes \cdots \otimes r_{n-1} \otimes r_n y) \end{aligned}$$

is defined for any $n \geq 1$ gives $\mathrm{Tor}_*^R(X,Y)$. Using [13, Lem. 2, pg. 248], we see that for any right R-module X, the module $X \otimes_S R$ is a (R,S)-projective module. This implies the relative two sided bar complex $\mathrm{CB}_*(X,R|S,Y)$ which is defined as $\bigoplus_{n\geq 0} X \otimes_S R^{\otimes_S n} \otimes_S Y$ with the differentials induced from the ordinary bar complex yields the relative Tor groups $\mathrm{Tor}_*^{(R,S)}(X,Y)$. More importantly, there is a comparison natural transformation between the derived functors

$$c_*^{X,Y}\colon \mathrm{Tor}_*^R(X,Y) \to \mathrm{Tor}_*^{(R,S)}(X,Y)$$

There are similar comparison morphisms between the derived functors

$$c^*_{X,Z}\colon \mathrm{Ext}^*_{(R,S)}(X,Z) \to \mathrm{Ext}^*_R(X,Z)$$

since $\mathrm{Ext}^*_{(R,S)}(X,Z) = H^*\mathrm{Hom}_R(\mathrm{CB}_*(X,R|S,R),Z)$ for an arbitrary pair of R-modules (X,Z) of the same parity. If S is a semi-simple k-algebra then the class of (R,S)-projective modules coincide with the class of R-projective modules because of Lemma 2.4. Then the comparison natural transformations are isomorphisms. □

3. Product Categories and Double Functors

Definition 3.1. Let $\mathcal{U}$ and $\mathcal{V}$ be two k-linear small categories. Define a new category $\mathcal{U}\otimes\mathcal{V}$ as follows. The set of objects of $\mathcal{U}\otimes\mathcal{V}$ are pairs of the form (U,V) with $U\in Ob(\mathcal{U})$ and $V\in Ob(\mathcal{V})$. Given two objects (U,V) and (U',V') in $Ob(\mathcal{U}\otimes\mathcal{V})$ we define

$$\mathrm{Hom}_{\mathcal{U}\otimes\mathcal{V}}((U,V),(U',V')) := \mathrm{Hom}_{\mathcal{U}}(U,U')\otimes\mathrm{Hom}_{\mathcal{V}}(V,V')$$

The compositions are defined as $(f\otimes f')\circ(g\otimes g') = f\circ g\otimes f'\circ g'$ if $f\otimes f'$ and $g\otimes g'$ are two composable morphisms. Note that we always have

$$f\otimes g = (f\otimes t(g))\circ(s(f)\otimes g) = (t(f)\otimes g)\circ(f\otimes s(g))$$

for any $f\in\mathrm{Hom}_{\mathcal{U}}$ and $g\in\mathrm{Hom}_{\mathcal{V}}$ where we use $s(\alpha)$ and $t(\alpha)$ to denote the source and the target of and morphism α. We also use the convention that the symbol for an object also denotes the identity morphism on the same object.

Remark 3.2. Note that when $\mathcal{U}$ and $\mathcal{V}$ are k-linear abelian categories, the product category $\mathcal{U}\otimes\mathcal{V}$ is a k-linear category, but not necessarily an abelian category. For each object $U\in Ob(\mathcal{U})$ and $V\in Ob(\mathcal{V})$ we have subcategories $\mathcal{U}\otimes V$ and $U\otimes\mathcal{V}$ of $\mathcal{U}\otimes\mathcal{V}$ which consists of objects

$$Ob(\mathcal{U}\otimes V) := \{(U,V)|\ U\in Ob(\mathcal{U})\} \quad\text{and}\quad Ob(U\otimes\mathcal{V}) := \{(U,V)|\ V\in Ob(\mathcal{V})\}$$

where the morphisms are

$$\mathrm{Hom}_{\mathcal{U}\otimes V}((U,V),(U',V)) = \mathrm{Hom}_{\mathcal{U}}(U,U')\otimes k\{id_V\}$$

and

$$\mathrm{Hom}_{U\otimes\mathcal{V}}((U,V),(U,V')) = k\{id_U\}\otimes\mathrm{Hom}_{\mathcal{U}}(V,V')$$

These subcategories are abelian.

Definition 3.3. A k-linear functor of the form $F\colon\mathcal{U}\otimes\mathcal{V}\to\mathcal{W}$ is called a k-linear double functor. Such a double functor is called exact (resp. left exact or right exact) if $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ are k-linear abelian categories, and the restriction functors

$$F_U\colon U\otimes\mathcal{V}\to\mathcal{W} \quad\text{and}\quad F_V\colon\mathcal{U}\otimes V\to\mathcal{W}$$

are both exact (resp. left exact or right exact) for any $U\in Ob(\mathcal{U})$ and $V\in Ob(\mathcal{V})$.

Definition 3.4. Assume $\mathcal{U}$ is an abelian category. Consider the category $\mathbf{Ch}(\mathcal{U})$ of differential $\mathbb{Z}$-graded objects in $\mathcal{U}$ with differentials of degree -1, and the graded morphisms of $\mathbb{Z}$-graded objects between these objects. This choice of morphisms makes each Hom set $\mathrm{Hom}_{\mathbf{Ch}(\mathcal{U})}(U_*, U'_*)$ into a differential graded k-module as follows: for any $n \in \mathbb{Z}$ we let

$$\mathrm{Hom}^n_{\mathbf{Ch}(\mathcal{U})}(U_*, U'_*) := \prod_{m\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{U}}(U_m, U'_{m+n})$$

The differentials on the Hom modules are given by

$$d^n_{\mathcal{U}}(f_*) := \left(d'_{m+n} \circ f_m + (-1)^{n+1} f_{m+n-1} \circ d_m\right)_{m\in\mathbb{Z}}$$

for any $f_* \in \mathrm{Hom}_{\mathbf{Ch}(\mathcal{U})}(U_*, U'_*)$ where d_* is the differential of U_* and d'_* is the differential of U'_*. One can easily show that the composition of morphisms is graded and the differentials $d^*_{\mathcal{U}}$ satisfy Leibniz rule with respect to compositions, i.e.

$$d^{i+j}_{\mathcal{U}}(f_* \circ g_*) = d^i_{\mathcal{U}}(f_*) \circ g_* + (-1)^i f_* \circ d^j_{\mathcal{U}}(g_*)$$

for a composable pair of morphisms f_* and g_* of degree i and j respectively. We observe that $ker(d^0_{\mathcal{U}})$ consists of morphisms of $\mathbb{Z}$-graded objects of degree 0 which commute with their differentials, and $im(d^1_{\mathcal{U}})$ consists of morphisms of differential graded objects which are null-homotopic. Thus using Hom sets of the form $H_0\mathrm{Hom}^*_{\mathbf{Ch}(\mathcal{U})}(U_*, U'_*)$ gives us the quotient of the category of differential graded objects and their degree 0 morphisms in $\mathcal{U}$ by the subcategory of null-homotopic morphisms. This category is usually denoted by $\mathbf{K}(\mathcal{U})$.

Lemma 3.5. *Assume $\mathcal{W}$ has countable products (i.e. limits over countable discrete categories), and that such products are exact. If $F\colon \mathcal{U}\otimes\mathcal{V} \to \mathcal{W}$ is a k-linear double functor then F induces a functor of the form $\mathbf{K}(F)\colon \mathbf{K}(\mathcal{U})\otimes\mathbf{K}(\mathcal{V}) \to \mathbf{K}(\mathcal{W})$ which is defined on the objects by*

$$\mathbf{K}(F)(U_*, V_*) := Tot^{\Pi}_* F(U_*, V_*)$$

for any $U_* \in Ob(\mathbf{Ch}(\mathcal{U}))$ and $V_* \in Ob(\mathbf{Ch}(\mathcal{V}))$.

Proof. First, let us describe $\mathbf{Ch}(\mathcal{U}) \otimes \mathbf{Ch}(\mathcal{V})$. Its objects are pairs of the form (U_*, V_*) where U_* and V_* are from $\mathbf{Ch}(\mathcal{U})$ and $\mathbf{Ch}(\mathcal{V})$ respectively. The bi-graded k-module of morphisms between two objects (U_*, V_*) and (U'_*, V'_*) is defined to be

$$\mathrm{Hom}^{i,j}_{\mathbf{Ch}(\mathcal{U})\otimes\mathbf{Ch}(\mathcal{V})}((U_*, V_*), (U'_*, V'_*)) := \prod_{p,q\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{U}\otimes\mathcal{V}}((U_p, V_q), (U'_{p+i}, V'_{q+j}))$$

$$:= \prod_{p,q\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{U}}(U_p, U'_{p+i}) \otimes \mathrm{Hom}_{\mathcal{V}}(V_q, V'_{q+j})$$

Now observe that $F(U_*, V_*)$ is also a bi-differential object in $\mathcal{W}$ with horizontal and vertical differentials for all $p, q \in \mathbb{Z}$ are defined as

$$d^h_{p,q} := F(d_p, V_q) \quad \text{and} \quad d^v_{p,q} := F(U_p, d_q)$$

It is easy to see that if $f_* \in \mathrm{Hom}^i_{\mathbf{Ch}(\mathcal{U})}(U_*, U'_*)$ and $g_* \in \mathrm{Hom}^j_{\mathbf{Ch}(\mathcal{V})}(V_*, V'_*)$ then

$$Tot^{\Pi}_* F(f_*, g_*) \in \mathrm{Hom}^{i+j}_{\mathbf{Ch}(\mathcal{W})}(Tot^{\Pi}_* F(U_*, V_*), Tot^{\Pi}_* F(U'_*, V'_*))$$

since $F(f_*, g_*) = F(f_*, V'_*) \circ F(U_*, g_*)$ and the composition of morphisms in $\mathbf{Ch}(\mathcal{W})$ is graded. Since the differential on the total complex $Tot^{\Pi}_* F(U_*, V_*)$ is defined as the sum $d^h_{*,*} + d^v_{*,*}$ we see that

$$d^{i+j}_{\mathcal{W}} Tot^{\Pi}_m F(f_*, g_*) := \left((d^h_{p+i,q+j} + d^v_{p+i,q+j}) F(f_p, g_q) + (-1)^{i+j+1} F(f_p, g_q)(d^v_{p+1,q} + d^h_{p,q+1}) \right)_{p+q=m}$$

where f_* and g_* are as before. One can reduce this formula further to

$$d^{i+j}_{\mathcal{W}} Tot^{\Pi}_* F(f_*, g_*) = Tot^{\Pi}_* F(d^i_{\mathcal{U}} f_*, g_*) + (-1)^i Tot^{\Pi}_* F(f_*, d^j_{\mathcal{V}} g_*)$$

In order to have a well-defined functor $\mathbf{K}(F)$, one must have the following conditions satisfied.

(i) If f_* is in $ker(d^0_{\mathcal{U}})$ and g_* is in $ker(d^0_{\mathcal{V}})$ then $Tot^{\Pi}_* F(f_*, g_*)$ is in $ker(d^0_{\mathcal{W}})$.

(ii) For any $a_* \in \mathrm{Hom}^1_{\mathbf{Ch}(\mathcal{U})}(U_*, U'_*)$, $c_* \in ker(d^0_{\mathcal{U}})$, $b_* \in ker(d^0_{\mathcal{V}})$ and $e_* \in \mathrm{Hom}^1_{\mathbf{Ch}(\mathcal{V})}(V_*, V'_*)$ one must have

$$Tot^{\Pi}_* F(d^1_{\mathcal{U}}(a_*), b_*) + Tot^{\Pi}_* F(c_*, d^1_{\mathcal{V}}(e_*))$$

in the image of $d^1_{\mathcal{W}}$.

In order to prove (i) we compute

$$d^0_{\mathcal{W}} Tot^{\Pi}_* F(f_*, g_*) = Tot^{\Pi}_* F(d^0_{\mathcal{U}} f_*, g_*) + Tot^{\Pi}_* F(f_*, d^0_{\mathcal{V}} g_*) = 0$$

For (ii) we observe

$$\begin{aligned} & d^1_{\mathcal{W}} \left(Tot^{\Pi}_* F(a_*, b_*) + Tot^{\Pi}_* F(c_*, e_*) \right) \\ =& Tot^{\Pi}_* F(d^1_{\mathcal{U}} a_*, b_*) - Tot^{\Pi}_* F(a_*, d^0_{\mathcal{V}} b_*) + Tot^{\Pi}_* F(d^0_{\mathcal{U}} c_*, e_*) + Tot^{\Pi}_* F(c_*, d^1_{\mathcal{V}} e_*) \\ =& Tot^{\Pi}_* F(d^1_{\mathcal{U}}(a_*), b_*) + Tot^{\Pi}_* F(c_*, d^1_{\mathcal{V}}(e_*)) \end{aligned}$$

as we wanted to prove. □

Remark 3.6. Given a functor between two small k-linear abelian categories $G\colon \mathcal{U} \to \mathcal{V}$, we will get a functor of the form $\mathbf{Ch}(G)\colon \mathbf{Ch}(\mathcal{U}) \to \mathbf{Ch}(\mathcal{V})$. Both $\mathbf{Ch}(\mathcal{U})$ and $\mathbf{Ch}(\mathcal{V})$ are categories enriched over the category of differential graded k-modules, i.e. they are differential graded k-linear categories and $\mathbf{Ch}(G)$ is a functor of differential graded k-linear categories [17]. Now, given a double functor $F\colon \mathcal{U} \otimes \mathcal{V} \to \mathcal{W}$, one can show that $\mathbf{Ch}(F)\colon \mathbf{Ch}(\mathcal{U}) \otimes \mathbf{Ch}(\mathcal{V}) \to \mathbf{Ch}(\mathcal{W})$ is a functor of differential graded k-linear categories with a little work. Since the domain is a product category which is normally enriched over the category of bi-differential bi-graded k-modules, the maps on the Hom k-modules need to be interpreted carefully using total complexes on the Hom k-modules. Then the result we prove above is equivalent to the existence of a functor of the form

$$H_{p,q}(F)\colon H_p\mathbf{Ch}(\mathcal{U}) \otimes H_q\mathbf{Ch}(\mathcal{V}) \to H_{p+q}\mathbf{Ch}(\mathcal{W})$$

for $(p,q) = (0,0)$, which actually holds for all (p,q).

Theorem 3.7. *Assume $\mathcal{W}$ has countable products and that such products are exact. If $F\colon \mathcal{U} \otimes \mathcal{V} \to \mathcal{W}$ is a k-linear exact double functor then $\mathbf{K}(F)$ extends to a functor on the product of the derived categories of the form $\mathbf{D}(F)\colon \mathbf{D}(\mathcal{U}) \otimes \mathbf{D}(\mathcal{V}) \to \mathbf{D}(\mathcal{W})$ and we obtain natural transformations of double functors*

$$\mathrm{Ext}^p_{\mathcal{U}}(\,\cdot\,, U) \otimes \mathrm{Ext}^q_{\mathcal{V}}(\,\cdot\,, V) \to \mathrm{Ext}^{p+q}_{\mathcal{W}}(F(\,\cdot\,,\,\cdot\,), F(U,V))$$

for any fixed pair of objects $U \in Ob(\mathcal{U})$ and $V \in Ob(\mathcal{V})$, compatible with the triangulated structures in both variables for any p and q.

Proof. We will show that quasi-isomorphisms in each variable, with the other variable fixed, are sent to quasi-isomorphisms in $\mathbf{Ch}(\mathcal{W})$ in order to get the required extension. We will prove this for the first variable. The proof for the second variable is identical. Assume $f_*\colon U_* \to U'_*$ is a quasi-isomorphism in $\mathbf{Ch}(\mathcal{U})$. By [24, Corollary 1.5.4] f_* is a quasi-isomorphism if and only if the mapping cone of f_*, here denoted by $\mathrm{Cone(f_*)}$, is acyclic and we have a split exact sequence of differential graded objects

$$0 \to U'_* \to \mathrm{Cone}(f_*) \to U_*[-1] \to 0$$

Since Tot^{Π}_* is an exact functor from $\mathbf{Ch}^2(\mathcal{W})$ the category of bi-differential bi-graded objects in $\mathcal{W}$ to $\mathbf{Ch}(\mathcal{W})$, and F is exact in the first variable, this short exact sequence translates to

$$0 \to Tot^{\Pi}_* F(U'_*, V_*) \to Tot^{\Pi}_* F(\mathrm{Cone}(f_*), V_*) \to Tot^{\Pi}_* F(U_*[-1], V_*) \to 0$$

Again, since F is exact, $F(\mathrm{Cone}(f_*), V_q)$ is acyclic for every $q \in \mathbb{Z}$. Then by a spectral sequence argument we conclude that the total complex $Tot^{\Pi}_* F(\mathrm{Cone}(f_*), V_*)$ is acyclic. Therefore $Tot^{\Pi}_* F(f_*, V_*)$ induces an isomorphism on homology for any V_* in $\mathbf{Ch}(\mathcal{V})$ using the resulting long exact sequence in homology. This finishes the proof of the first part. The subcategories that we are interested in the derived categories $\mathbf{D}(\mathcal{U})$, $\mathbf{D}(\mathcal{V})$ and $\mathbf{D}(\mathcal{W})$ are the full subcategories of the complexes whose homologies are concentrated at only one degree. Since F is exact, $\mathbf{D}(F)$ sends the objects of the form $U[i]$ in $\mathbf{D}(\mathcal{U})$ and $V[j]$ in $\mathbf{D}(\mathcal{V})$ to an object of the form $F(U,V)[i+j]$ in $\mathbf{D}(\mathcal{W})$. Here U, V and $F(U,V)$ are ordinary objects in $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ respectively, and we use the notation that for an ordinary object A in an abelian category $\mathcal{A}$ the symbol $A[n]$ represents an object in the derived category of $\mathcal{A}$ which is the same A considered as a complex with 0 differentials and concentrated at degree $n \in \mathbb{Z}$. The result follows. $\square$

Remark 3.8. The natural transformation of double functors we obtained in Theorem 3.7 is actually a natural transformation of quadruple functors if we do not fix two of the variables. But we will need the result in this form.

Lemma 3.9. *Assume $G^*\colon \mathcal{U}\otimes\mathcal{V} \to k$-**Mod** is a graded double functor which is a cohomological δ-functor* [24, Def. 2.1.1] *in each variable. Fix $U \in Ob(\mathcal{U})$ and $V \in Ob(\mathcal{V})$. Assume also that we have natural transformations of double functors of the form*

$$\eta^{p,q}\colon \mathrm{Ext}^p_{\mathcal{U}}(\,\cdot\,, U) \otimes \mathrm{Ext}^q_{\mathcal{V}}(\,\cdot\,, V) \to G^{p+q}(\,\cdot\,,\,\cdot\,)$$

compatible with the δ-structures in both of the variables for any $p, q \geq 0$. If there exists an $n \in \mathbb{N}$ such that $\eta^{p,q} = 0$ for any $p+q = n$ then $\eta^{p,q} = 0$ for any $p+q \geq n$.

Proof. Fix another object $V' \in Ob(\mathcal{V})$ and consider a short exact sequence $0 \leftarrow U'' \leftarrow P \leftarrow U' \leftarrow 0$ in $\mathcal{U}$ where P is projective. We obtain a commutative diagram of the form

$$\begin{tikzcd}
\mathrm{Ext}^p_{\mathcal{U}}(U',U)\otimes\mathrm{Ext}^q_{\mathcal{V}}(V',V) \arrow[r, "\delta^p\otimes id"] \arrow[d, "\eta^{p,q}"', "0"] & \mathrm{Ext}^{p+1}_{\mathcal{U}}(U'',U)\otimes\mathrm{Ext}^q_{\mathcal{V}}(V',V) \arrow[r, "0"] \arrow[d, "\eta^{p+1,q}"'] & \mathrm{Ext}^{p+1}_{\mathcal{U}}(P,U)\otimes\mathrm{Ext}^q_{\mathcal{V}}(V',V) \arrow[d, "\eta^{p+1,q}"] \\
G^{p+q}(U',V') \arrow[r] & G^{p+q+1}(U'',V') \arrow[r] & G^{p+q+1}(P,V')
\end{tikzcd}$$

Since P is projective $\operatorname{Ext}_{\mathcal{U}}^{p+1}(P,U)=0$ and $\delta^p\otimes id$ is an epimorphism for any $p\geq 0$. On the other hand $\eta^{p+1,q}(\delta^p\otimes id)=0$ since $\eta^{p,q}=0$. Because $\delta^p\otimes id$ is an epimorphism, we have $\eta^{p+1,q}=0$. One can repeat the same argument for the other variable. Since U'' was arbitrary and short exact sequences of the form $0\leftarrow U''\leftarrow P\leftarrow U'\leftarrow 0$ always exist because $\mathcal{U}$ (and $\mathcal{V}$ for the second variable) have enough projectives, the result follows. $\square$

4. Diagram Modules and Pairings

Definition 4.1. Let $\mathbf{C}$ be a small category and let $\mathbf{C}$-**Mod** (resp. **Mod**-$\mathbf{C}$) denote the category of covariant (resp. contravariant) functors from $\mathbf{C}$ to k-**Mod** and their natural transformations. We will call such functors left (resp. right) $\mathbf{C}$-modules. Such a module $X_\bullet$ is a direct sum of k-modules indexed by the set of objects of $\mathbf{C}$ of the form $\bigoplus_{a\in Ob(\mathbf{C})} X_a$. The primary examples we have in mind are $\mathbf{C}=\Delta$ the simplicial category, or $\mathbf{C}=\Lambda$ Connes' cyclic category. We will use the following notation

$$(a\xleftarrow{f} b)\triangleright x := X_f(x) \qquad \text{(resp. } x\triangleleft (a\xleftarrow{f} b) := X_f(x))$$

for any $x\in X_a$ (resp. $x\in X_b$) where $X_f\colon X(b)\to X(a)$ is the evaluation of $X_\bullet$ on the morphism $a\xleftarrow{f} b$ in $\mathbf{C}$.

Remark 4.2. In order to simplify notation, for a small category $\mathbf{C}$ we will use the notation $\operatorname{Hom}_{\mathbf{C}}$ for morphisms of $\mathbf{C}$-modules. We will extend this simplification to the derived functors as well and use $\operatorname{Ext}^*_{\mathbf{C}}$ for the derived functors of the double functor $\operatorname{Hom}_{\mathbf{C}}$ for both left and right $\mathbf{C}$-modules.

Definition 4.3. We define the $\mathbf{C}$-module $k_\bullet$ by letting $k_a=k$ for any $a\in Ob(\mathbf{C})$ and we let

$$1_a\triangleleft (a\xleftarrow{f} b)=1_b \quad \text{or} \quad (a\xleftarrow{f} b)\triangleright 1_b = 1_a$$

for any $f\colon b\to a$ in $\mathbf{C}$ depending on whether we view it as a left or right $\mathbf{C}$-module.

Definition 4.4. Assume $X_\bullet$ is a left $\mathbf{C}$-module and $Y_\bullet$ is a right $\mathbf{C}$-module. Let

$$diag_\bullet \operatorname{Hom}_k(X_\bullet,Y_\bullet) := \bigoplus_{a\in Ob(\mathbf{C})} \operatorname{Hom}_k(X_a,Y_a)$$

By definition $diag_\bullet \operatorname{Hom}_k(X_\bullet,Y_\bullet)$ is a k-module indexed by the set of objects of $\mathbf{C}$. Also, given any $\psi\in diag_a \operatorname{Hom}_k(X_\bullet,Y_\bullet)$ and $f\in \operatorname{Hom}_{\mathbf{C}}(b,a)$ define

$$(\psi\cdot (a\xleftarrow{f} b))(x) := \psi((a\xleftarrow{f} b)\triangleright x)\triangleleft (a\xleftarrow{f} b)\in diag_b \operatorname{Hom}_k(X_\bullet,Y_\bullet)$$

for any $x\in X_b$. So, $\psi\cdot(a\xleftarrow{f} b)$ is in $diag_b \operatorname{Hom}_k(X_\bullet,Y_\bullet)$

Proposition 4.5. *The assignment $diag_\bullet \operatorname{Hom}_k(\,\cdot\,,\,\cdot\,)$ defines an exact double functor of the form*

$$diag_\bullet \operatorname{Hom}_k(\,\cdot\,,\,\cdot\,)\colon (\textbf{Mod-}\mathbf{C})^{op}\otimes \mathbf{C}\textbf{-Mod}\to \mathbf{C}\textbf{-Mod}$$

which induces a natural transformation of double functors

$$(\,\cdot\smile\cdot\,)\colon \operatorname{Ext}^p_{\mathbf{C}}(k_\bullet,\,\cdot\,)\otimes \operatorname{Ext}^q_{\mathbf{C}}(\,\cdot\,,k_\bullet)\to \operatorname{Ext}^{p+q}_{\mathbf{C}}(diag_\bullet \operatorname{Hom}_k(\,\cdot\,,\,\cdot\,),k_\bullet)$$

for any $p,q\geq 0$.

Proof. First, we must show that the action of $\mathbf{C}$ on $diag_\bullet \mathrm{Hom}_k(X_\bullet, Y_\bullet)$ is associative for any left $\mathbf{C}$-module $X_\bullet$ and right $\mathbf{C}$-module $Y_\bullet$. We observe

$$\begin{aligned}((\psi\cdot(a\xleftarrow{f}b))\cdot(b\xleftarrow{g}c))(x) =&(\psi\cdot(a\xleftarrow{f}b))((b\xleftarrow{g}c)\triangleright x)\triangleleft(b\xleftarrow{g}c)\\ =&(\psi((a\xleftarrow{f}b)\triangleright((b\xleftarrow{g}c)\triangleright x))\triangleleft(a\xleftarrow{f}b))\triangleleft(b\xleftarrow{g}c)\\ =&\psi((a\xleftarrow{fg}c)\triangleright x)\triangleleft(a\xleftarrow{fg}c)\\ =&(\psi\cdot(a\xleftarrow{fg}c))(x)\end{aligned}$$

for any $x \in diag_c \mathrm{Hom}_k(X_\bullet, Y_\bullet)$ and $f\colon b\to a$ and $g\colon c\to b$ in $\mathbf{C}$. The exactness of the double functor follows form the fact that k is a field. Now, observe that $diag_\bullet \mathrm{Hom}_k(k_\bullet, k_\bullet)\cong k_\bullet$ apply Theorem 3.7 to get the prescribed natural transformation. ☐

Remark 4.6. One has to be careful in interpreting the derivatives of the double functor and the pairing we obtained in Proposition 4.5 in cases where one would like to use bounded above or bounded below derived categories. In such, cases we have

$$\mathbf{D}_+(diag_\bullet \mathrm{Hom}_k)\colon \mathbf{D}_-(\mathbf{C}\textbf{-Mod})\otimes\mathbf{D}_+(\textbf{Mod-}\mathbf{C})\to\mathbf{D}_+(\textbf{Mod-}\mathbf{C})$$

or

$$\mathbf{D}_-(diag_\bullet \mathrm{Hom}_k)\colon \mathbf{D}_+(\mathbf{C}\textbf{-Mod})\otimes\mathbf{D}_-(\textbf{Mod-}\mathbf{C})\to\mathbf{D}_-(\textbf{Mod-}\mathbf{C})$$

because our functor $diag_\bullet \mathrm{Hom}_k$ is contravariant in the first variable.

Remark 4.7. For the curious reader who would like to see an explicit formula for the pairing we defined above, we note that the pairing is a slight modification of the external product in cohomology. So, the Alexander-Whitney map (cf. [19, Thm. 8.5] or [24, 8.5.4]), applied correctly is going to work. We ask the reader to pick his/her favorite cosimplicial module $\mathcal{X}_{\bullet,\bullet}$ which consists of injective $\mathbf{C}$-modules whose (singular) homology is the $\mathbf{C}$-module $X_\bullet$ concentrated at degree 0, and simplicial module $\mathcal{Y}_{\bullet,\bullet}$ which consists of projective $\mathbf{C}$-modules whose (singular) homology is the $\mathbf{C}$-module $Y_\bullet$ concentrated at degree 0. Such modules exist because of Dold-Kan equivalence [9]. Then for two given cochains $\xi\colon k_\bullet\to\mathcal{X}_{p.\bullet}$ and $\nu\colon \mathcal{Y}_{q,\bullet}\to k_\bullet$ we define a new cochain $\xi\smile\nu\colon diag_\bullet \mathrm{Hom}_k(\mathcal{X}_{p+q,\bullet},\mathcal{Y}_{p+q,\bullet})\to k_\bullet$ by

$$(\xi\smile\nu)(\eta) := \nu\circ\underbrace{\partial^{\mathcal{Y}}_{q+2}\circ\cdots\partial^{\mathcal{Y}}_{p+q+1}}_{p\text{-terms}}\circ\eta\circ\underbrace{\partial^{\mathcal{X}}_{p+q}\circ\cdots\circ\partial^{\mathcal{X}}_{p+1}}_{q\text{-terms}}\circ\xi$$

for any $\eta\in diag_\bullet \mathrm{Hom}_k(\mathcal{X}_{p+q,\bullet},\mathcal{Y}_{p+q,\bullet})$ where we use $\partial^{\mathcal{Z}}_i$ to denote the (co)face maps of a (co)simplicial object $\mathcal{Z}_\bullet$.

5. CYCLIC (CO)HOMOLOGY

Definition 5.1. Let Λ also denote the k-algebra generated by the arrows of Connes' cyclic category Λ. Here we will give a specific presentation of the k-algebra Λ. We will denote the generators by $\partial^n_j\colon [n]\to$

$[n+1]$, $\sigma_i^n\colon [n+1] \to [n]$ and $\tau_n^\ell\colon [n] \to [n]$ for any $[n] \in Ob(\Lambda)$ with $0 \le j \le n+1$, $0 \le i \le n$ and $0 \le \ell \le n$. The relations are

$$\begin{aligned} \partial_i^{n+1}\partial_j^n &= \partial_{j+1}^{n+1}\partial_i^n \text{ for } i \le j & \sigma_j^n\sigma_i^{n+1} &= \sigma_i^n\sigma_{j+1}^{n+1} \text{ for } i \le j \\ \partial_i^n\sigma_j^n &= \sigma_{j+1}^{n+1}\partial_i^{n+1} \text{ for } i \le j & \partial_i^n\sigma_j^n &= \sigma_j^{n+1}\partial_{i+1}^{n+1} \text{ for } i > j \\ \sigma_i^n\partial_i^n &= id_i = \sigma_i^n\partial_{i+1}^n & \tau_n^\ell\tau_n &= \tau_n^{\ell+1} \text{ and } \tau_n^{n+1} = id_n \\ \partial_j^n\tau_n^i &= \tau_{n+1}^{i+p}\partial_q^n \text{ for } i+j = (n+1)p+q & \sigma_j^n\tau_{n+1}^i &= \tau_n^{i-p}\sigma_q^n \text{ for } i+j = (n+1)p+q \end{aligned}$$

All other products between the generators are 0. Note that Λ is not a unital k-algebra, but a H-unital algebra [25]. One also can view Λ first as a bimodule then as an algebra over its subalgebra $\mathcal{K} := \bigoplus_{n\ge 0} k\{id_n\}$ generated by id_n for any $n \ge 0$.

Remark 5.2. Let $\mathbb{F}$ be the set of all finite subsets of $\mathbb{N}$ the set of all natural numbers. Note that Λ is the colimit of unital k-algebras $\operatorname{colim}_{U\in\mathbb{F}}\Lambda_U$ where for an arbitrary $U \in \mathbb{F}$ the k-algebra Λ_U is the unital subalgebra of Λ generated by elements Ψ which satisfy the property that $\Psi = id_m\Psi id_n$ for some $m, n \in U$. Thus Λ is H-unital. Because of this property, we are interested only in *locally finite and faithful modules* over Λ. These are graded k-modules $M_* = \bigoplus_{n\ge 0} M_n$ such that $id_n m_n = m_n$ for any $m_n \in M_n$. These modules have the property that they can be written as a colimit of unital modules over the unital algebras Λ_U for $U \in \mathbb{F}$. Thus one can prove statements for the unital algebras Λ_U for $U \in \mathbb{F}$ and their modules $M_U = \operatorname{Res}^\Lambda_{\Lambda_U} M$ then lift the argument to Λ and M by using a colimit taken over $U \in \mathbb{F}$.

Definition 5.3. Let $\mathcal{X} = \bigoplus_{n,m\in\mathbb{N}} X_{n,m}$ be a $\mathcal{K}$-bimodule. Assume $\mathcal{A}$ is a $\mathcal{K}$-algebra. An isomorphism of $\mathcal{K}$-bimodules $\omega_{\mathcal{X}}\colon \mathcal{X}\otimes_{\mathcal{K}}\mathcal{A} \to \mathcal{A}\otimes_{\mathcal{K}}\mathcal{X}$ is called a transposition if (i) one has a commutative diagram of the form

$$\begin{tikzcd} \mathcal{A}\otimes_{\mathcal{K}}\mathcal{A}\otimes_{\mathcal{K}}\mathcal{X} \arrow[r, "\mathcal{A}\otimes\omega_{\mathcal{X}}"] \arrow[d, "\mu\otimes\mathcal{X}"'] & \mathcal{A}\otimes_{\mathcal{K}}\mathcal{X}\otimes_{\mathcal{K}}\mathcal{A} \arrow[r, "\omega_{\mathcal{X}}\otimes\mathcal{A}"] & \mathcal{X}\otimes_{\mathcal{K}}\mathcal{A}\otimes_{\mathcal{K}}\mathcal{A} \arrow[d, "\mathcal{X}\otimes\mu"] \\ \mathcal{A}\otimes_{\mathcal{K}}\mathcal{X} \arrow[rr, "\omega_{\mathcal{X}}"'] & & \mathcal{X}\otimes_{\mathcal{K}}\mathcal{A} \end{tikzcd}$$

where $\mu\colon \mathcal{A}\otimes_{\mathcal{K}}\mathcal{A} \to \mathcal{A}$ is the multiplication structure on $\mathcal{A}$, and (ii) we have $\omega(1_p \otimes x) = (x \otimes 1_q)$ for any $x \in X_{p,q}$ and $p, q \in \mathbb{N}$.

Remark 5.4. Consider Δ the subalgebra of Λ generated by elements ∂_j^n, σ_i^n and id_n for any $n \in \mathbb{N}$ and all possible i, j; and the subalgebra $\mathbf{T}$ of Λ generated by τ_n^i for all possible $n \ge 0$ and $0 \le i \le n$. Define a transposition $\omega\colon \Delta\otimes_{\mathcal{K}}\mathbf{T} \to \mathbf{T}\otimes_{\mathcal{K}}\Delta$ using the relations in Λ

$$\begin{aligned} \omega(\partial_j^n \otimes \tau_n^i) =& \tau_{n+1}^{i+p}\otimes\partial_q^n \text{ with } (i+j) = (n+1)p+q \text{ and } 0 \le q \le n \\ \omega(\sigma_s^n \otimes \tau_{n+1}^s) =& \tau_n^{s-a}\otimes\sigma_b^n \text{ with } (s+t) = (n+2)a+b \text{ and } 0 \le b \le n+1 \end{aligned}$$

It is easy to see that ω is invertible and the inverse is given by

$$\begin{aligned} \omega^{-1}(\tau_{n+1}^i \otimes \partial_j^n) =& \partial_q^n \otimes \tau_n^{i-p} \text{ with } (-i+j) = (n+2)(-p)+q \text{ and } 0 \le q \le n+1 \\ \omega^{-1}(\tau_n^s \otimes \sigma_t^n) =& \sigma_b^n \otimes \tau_{n+1}^{s-a} \text{ with } (-s+t) = (n+1)(-a)+b \text{ and } 0 \le b \le n \end{aligned}$$

It is tedious but easy to show that ω and ω^{-1} are transpositions.

Remark 5.5. We can even split the subalgebra Δ of Λ into 2 pieces using an appropriate distributivity law as follows. Let $\mathcal{F}$ be the subalgebra of Λ generated by ∂_i^n and id_n for all $n \geq 0$ and $0 \leq i \leq n+1$. Let $\mathcal{D}$ be the subalgebra of Λ generated by all σ_i^n and id_n for all $n \geq 0$ and $0 \leq i \leq n$. There is a distributivity law of the form $\zeta\colon \mathcal{F} \otimes_{\mathcal{K}} \mathcal{D} \to \mathcal{D} \otimes_{\mathcal{K}} \mathcal{F}$ coming from the relations in Λ. Note that this distributivity law is not an isomorphism because of the relation $\sigma_i^n \partial_i^n = id_n = \sigma_i^n \partial_{i+1}^n$ for all $0 \leq i \leq n$ and $n \geq 0$.

Definition 5.6. Let CH_n be the left ideal of Δ generated by id_n. Define $d_n^{\mathrm{CH}}\colon \mathrm{CH}_{n+1} \to \mathrm{CH}_n$ by using the elements

$$d_n^{\mathrm{CH}} = \sum_{j=0}^{n+1} (-1)^j \partial_j^n$$

via right multiplication for any $n \geq 0$. One can easily see that $d_{n+1}^{\mathrm{CH}} d_n^{\mathrm{CH}} = 0$ for any $n \geq 0$.

Lemma 5.7. *The differential graded Δ-module $(\mathrm{CH}_*, d_*^{\mathrm{CH}})$ is a Δ-projective resolution of the trivial left Δ-module $k_\bullet$.*

Proof. The proof we present here is the same as that of [1, Lem. 2]. Notice that the left ideal $\langle id_n|$ of Δ generated by id_n is the free k-module on the set $\bigsqcup_m \mathrm{Hom}_\Delta(n, m)$ where, by abuse of notation, Δ denotes the category of finite ordinals and their order preserving maps. We observe that the arrows of this category generates our algebra Δ over k and the action of the algebra Δ is defined by pre-compositions on this set of generators. In other words, for each fixed m the differential graded k-module $id_m \mathrm{CH}_*$ computes the simplicial homotopy of the simplicial k-module $k[\Delta^m] := k[\mathrm{Hom}_\Delta(\bullet, m)]$ which is 0 everywhere except at degree 0, and is the ground field at degree 0. $\square$

Definition 5.8. Let $\mathcal{X}$ be a $\mathcal{K}$-bimodule and $\mathcal{A}$ be an augmented $\mathcal{K}$-algebra with augmentation $\epsilon\colon \mathcal{A} \to \mathcal{K}$. Let $\omega_{\mathcal{X}}\colon \mathcal{A} \otimes_{\mathcal{K}} \mathcal{X} \to \mathcal{X} \otimes_{\mathcal{K}} \mathcal{A}$ be a transposition. Then $\mathcal{X}$ carries a left $\mathcal{A}$-module structure $\lambda_{\mathcal{X}}$ which is defined as $(\mathcal{X} \otimes_{\mathcal{K}} \epsilon)\omega_{\mathcal{X}}$.

Definition 5.9. Let $\mathcal{X}$ and $\omega_{\mathcal{X}}$ be as before. Let $\mathcal{Y} = \bigoplus_{n,m\in\mathbb{N}} Y_{n,m}$ be another $\mathcal{K}$-bimodule and assume we have another transposition $\omega_{\mathcal{Y}}\colon \mathcal{A} \otimes_{\mathcal{K}} \mathcal{Y} \to \mathcal{Y} \otimes_{\mathcal{K}} \mathcal{A}$. Then the product $\mathcal{X} \otimes_{\mathcal{K}} \mathcal{Y}$ carries a left $\mathcal{A}$-module structure which is denoted by $\mathcal{X} \odot \mathcal{Y}$. The $\mathcal{A}$-module structure on $\mathcal{X} \odot \mathcal{Y}$ comes from the product transposition $\omega_{\mathcal{X}\odot\mathcal{Y}}\colon \mathcal{A} \otimes_{\mathcal{K}} \mathcal{X} \otimes_{\mathcal{K}} \mathcal{Y} \to \mathcal{X} \otimes_{\mathcal{K}} \mathcal{Y} \otimes_{\mathcal{K}} \mathcal{A}$ and the augmentation $\epsilon\colon \mathcal{A} \to \mathcal{K}$. The product transposition is defined as

$$\omega_{\mathcal{X}\odot\mathcal{Y}} := (\mathcal{X} \otimes \omega_{\mathcal{Y}}) \circ (\omega_{\mathcal{X}} \otimes \mathcal{Y})$$

and we let the left $\mathcal{A}$-module structure $\lambda_{\mathcal{X}\odot\mathcal{Y}}\colon \mathcal{A} \otimes_{\mathcal{K}} (\mathcal{X} \odot \mathcal{Y}) \to \mathcal{X} \odot \mathcal{Y}$ by

$$\lambda_{\mathcal{X}\odot\mathcal{Y}} := ((\mathcal{X} \odot \mathcal{Y}) \otimes \epsilon) \circ \omega_{\mathcal{X}\odot\mathcal{Y}}$$

Remark 5.10. Any $\mathbf{T}$-module $X_\bullet := \bigoplus_{n\geq 0} X_n$ admits a canonical transposition $\omega_X\colon \mathbf{T} \otimes_{\mathcal{K}} X_\bullet \to X_\bullet \otimes_{\mathcal{K}} \mathbf{T}$ which is defined as

$$\omega_X(\tau_n^\ell \otimes x) = \tau_n^\ell \cdot x \otimes \tau_n^\ell$$

for every $n \in \mathbb{N}$, $x \in X_n$ and $\ell \in \mathbb{Z}$.

Lemma 5.11. *Let $X_\bullet$ and $Y_\bullet$ be two $\mathbf{T}$-modules. Then one has an isomorphism of k-modules of the form*

$$X_\bullet \otimes_{\mathbf{T}} Y_\bullet \cong k_\bullet \otimes_{\mathbf{T}} (X_\bullet \odot Y_\bullet)$$

Proof. We are going to view $X_\bullet = \bigoplus_{n\in\mathbb{N}} X_n$ as a left $\mathbf{T}$-module via the action $\tau_n \cdot x := x \cdot \tau_n^{-\ell}$ for any $x \in X_n$ and $\ell \in \mathbb{Z}$. Since

$$X_\bullet \otimes_{\mathbf{T}} Y_\bullet = \bigoplus_{n\in\mathbb{N}} X_n \otimes_{\mathbb{Z}/(n+1)} Y_n$$

our statement reduces to proving $X \otimes_G Y \cong k \otimes_G (X \odot Y)$ where G is a finite abelian group, X is a right G-module, Y is a left G-module and $X \odot Y$ is the diagonal G-module $g\cdot(x\otimes y) = g\cdot x \otimes g \cdot y := x\cdot g^{-1} \otimes g\cdot y$ with $g \in G$, $x \in X$ and $y \in Y$. □

Theorem 5.12. *Let $X_\bullet$ be a right Λ-module (i.e. a cyclic module) and $Y_\bullet$ be a left Λ-module (i.e. a cocyclic module). Let $\mathrm{CH}^\lambda_*(X_\bullet)$ be the cyclic co-invariant quotient complex of the Hochschild complex of $X_\bullet$ and $\mathrm{CH}_\lambda^*(Y_\bullet)$ be the cyclic invariant subcomplex of the Hochschild complex of $Y_\bullet$. Then*

$$\mathrm{Tor}_*^{(\Lambda,\mathbf{T})}(X_\bullet, k_\bullet) \cong HC_*^\lambda(X_\bullet) \quad \textit{and} \quad \mathrm{Ext}^*_{(\Lambda,\mathbf{T})}(k_\bullet, Y_\bullet) \cong HC_\lambda^*(Y_\bullet)$$

where $HC_*^\lambda(X_\bullet)$ and $HC_\lambda^*(Y_\bullet)$ are the homologies of the complexes $\mathrm{CH}^\lambda_*(X_\bullet)$ and $\mathrm{CH}_\lambda^*(Y_\bullet)$, respectively.

Proof. Observe that we have a basis for Λ which consists of elements of the form

$$\sigma_{i_m}^m \cdots \sigma_{i_n}^n \partial_{j_n}^n \cdots \partial_{j_\ell}^\ell \tau_\ell^a \quad \text{where } i_m < \cdots < i_n \text{ and } j_n > \cdots > j_\ell$$

Using the transpositions ω and ω^{-1} we defined in Remark 5.4 we see that

$$\mathrm{CB}_n(\Lambda, \Lambda|\mathbf{T}, k_\bullet) = \overbrace{\Lambda \otimes_{\mathbf{T}} \cdots \otimes_{\mathbf{T}} \Lambda}^{n+1\text{-times}} \otimes_{\mathbf{T}} k_\bullet \cong \overbrace{\Delta \odot \cdots \odot \Delta}^{n+1\text{-times}} \odot k_\bullet = \mathrm{CB}_n(\Delta, \Delta, k_\bullet)$$

The left Δ-module structure on $\Delta^{\odot n+1} \odot k_\bullet$ comes from the left regular representation of Δ on itself on the left-most tensor component. The $\mathbf{T}$-module structure comes from the diagonal $\mathbf{T}$-module structure as defined in Definition 5.9 coming from the transposition ω^{-1} defined in Remark 5.4. Since $\mathbf{T}$ is semi-simple, the resolution $\mathrm{CB}_*(\Delta, \Delta, k_\bullet)$ can be replaced by the differential Δ-module CH_* which is also a left Λ-module structure coming from the transposition ω^{-1}. Then, the two sided bar complex $\mathrm{CB}_*(X_\bullet, \Lambda|\mathbf{T}, k_\bullet)$ can be replaced by

$$X_\bullet \otimes_\Lambda \mathrm{CH}_* \cong k_\bullet \otimes_{\mathbf{T}} (X_\bullet \otimes_\Delta \mathrm{CH}_*) = \mathrm{CH}^\lambda_*(X_\bullet)$$

The proof for $\mathrm{Ext}^*_{(\Lambda,\mathbf{T})}(k_\bullet, Y_\bullet)$ is similar. □

Proposition 5.13. *We have the natural isomorphisms of derived functors*

$$c_*^{\cdot\,,\,\cdot} : \mathrm{Tor}_*^\Lambda(\,\cdot\,,\,\cdot\,) \to \mathrm{Tor}_*^{(\Lambda,\mathbf{T})}(\,\cdot\,,\,\cdot\,) \qquad c^*_{\cdot\,,\,\cdot} : \mathrm{Ext}^*_{(\Lambda,\mathbf{T})}(\,\cdot\,,\,\cdot\,) \to \mathrm{Ext}^*_\Lambda(\,\cdot\,,\,\cdot\,)$$

Proof. We observe that $\mathbf{T}$ is a semi-simple subalgebra of Λ since we assume $char(k) = 0$ throughout. Now we use Proposition 2.5 and Remark 5.2. □

6. PAIRINGS IN CYCLIC (CO)HOMOLOGY

In this section we will use the notation and the terminology of [15] and [16] with the simplification that we use the same $k_\bullet$ for the trivial left and right Λ-module. In particular, $C_\bullet(Z, M)$ will denote the (co)cyclic module (i.e. Λ-module) associated with a H-module (co)algebra Z with coefficients in an arbitrary H-module/comodule M. Since the category of Λ-modules is abelian, Hopf-cyclic (co)homology of (co)cyclic modules are specific derived functors on this category

$$HC^*_{\rm Hopf}(A, M) = \mathrm{Ext}^*_\Lambda(C_\bullet(A, M), k_\bullet) \quad \text{and} \quad HC^*_{\rm Hopf}(C, M) = \mathrm{Ext}^*_\Lambda(k_\bullet, C_\bullet(C, M))$$

for an arbitrary H-module algebra A and H-module coalgebra C.

We recall the following definition from [15, Def. 2.2] to fix notation: C is said to act on A if there is a morphism of k-modules $\triangleright\colon C \otimes A \to A$ which satisfies (i) $c \triangleright (a_1 a_2) = (c^{(1)} \triangleright a_1)(c^{(2)} \triangleright a_2)$ and (ii) $c \triangleright 1_A = \varepsilon(c)1_A$ for any $a_1, a_2 \in A$ and $c \in C$. The action is called H-equivariant if $h(c \triangleright a) = h(c) \triangleright a$ for any $h \in H$, $a \in A$ and $c \in C$ where we use $h(c)$ to denote the action of H on the module (co)algebra C.

We obtained the following result in [15, Prop. 2.7].

Lemma 6.1. *Assume C acts on A equivariantly. The morphism of graded k-modules*

$$\phi_\bullet\colon Cyc_\bullet(A) \to diag_\bullet \mathrm{Hom}_k(C_\bullet(C, M), C_\bullet(A, M))$$

defined for $a_0 \otimes \cdots \otimes a_n \otimes m \in C_n(A, M)$ and $c_0 \otimes \cdots \otimes c_n \otimes m \in C_n(C, M)$ for any $n \geq 0$ by

$$\phi_n(a_0 \otimes \cdots \otimes a_n)(c_0 \otimes \cdots \otimes c_n \otimes m) = c_0 \triangleright a_0 \otimes \cdots \otimes c_n \triangleright a_n \otimes m$$

is a morphism of cyclic modules.

Theorem 6.2. *The morphism of cyclic modules $\phi_\bullet$ we defined in Lemma 6.1 induces a pairing of the form*

$$(\,\cdot\,\smile\,\cdot\,)\colon HC^p_{\rm Hopf}(C, M) \otimes HC^q_{\rm Hopf}(A, M) \to HC^{p+q}(A)$$

for any $p, q \geq 0$ where we use $HC^*_{\rm Hopf}$ to denote Hopf-cyclic cohomology and HC^* to denote the ordinary cyclic cohomology functors.

Proof. The pairing comes from Proposition 4.5 followed by Lemma 6.1. □

Remark 6.3. Theorem 6.2 gives us a pairing defined in $\mathbf{D}(\Lambda\text{-}\mathbf{Mod})$ the derived category of Λ-modules. Note that the pairing we obtain in Proposition 4.5 for the case $\mathbf{C} = \Lambda$ Connes' cyclic category, can easily be obtained in the relative derived category of cyclic modules $\mathbf{D}((\Lambda, \mathbf{T})\text{-}\mathbf{Mod})$, and with some work [15, Lem. 5.2, Lem. 5.3] also in $\mathbf{D}(\mathcal{M}\text{-}\mathbf{Mod})$ the derived category of mixed complexes. Thus followed by the induced map of $\phi_\bullet$ in cohomology we obtain similar pairings defined in $\mathbf{D}((\Lambda, \mathbf{T})\text{-}\mathbf{Mod})$ and $\mathbf{D}(\mathcal{M}\text{-}\mathbf{Mod})$. In [15, Thm. 5.4] we showed that the pairing we construct here and the pairing constructed in the derived category of mixed complexes are naturally isomorphic. Now, Proposition 5.13 gives us the natural isomorphism between the pairing we construct here and the pairing constructed in the relative derived category of cyclic modules.

Our aim is to show that pairings defined in [4, 10, 7, 18, 15, 23, 22] in Hopf-cyclic cohomology are naturally isomorphic as natural transformations of double functors. There are certain variations between

these pairings: The Connes-Moscovici, Gorokhovsky and Crainic pairings are defined for $C = H$, $q = 0$ and only for $M = k_{\sigma,\delta}$ the canonical 1-dimensional SAYD module associated with a modular pair involution in H. The Rangipour-Khalkhali pairing, and the pairing Rangipour defined in [22], are defined for an arbitrary module coalgebra C acting equivariantly on a module algebra A, and for arbitrary bi-degree (p, q) with an arbitrary SAYD module as coefficients [12]. Finally, the pairing we defined in [15] and here in Theorem 6.2 work in the same setup as the Khalkhali-Rangipour and Rangipour pairings but we allow arbitrary coefficient module/comodules.

The original pairing in Hopf-cyclic cohomology as defined by Connes and Moscovici [4] is constructed on the (co)cyclic module level, and (b, B)-complex is utilized to compute the Hopf-cyclic classes used as its input. This is done in the derived the category of mixed complexes [14]. Crainic, and later Nikonov and Sharygin defined their version of the pairing using Cuntz-Quillen formalism of X-complexes [8]. This is done in the homotopy category of towers of super complexes which is homotopy equivalent to the derived category of mixed complexes by Quillen [21]. In their setup Gorokhovsky [10], and later Khalkhali and Rangipour [18] also used mixed complexes to obtain their cohomology classes, and (H-invariant) closed graded (co)traces to implement their pairings. This is akin to Connes' use of closed graded traces to implement the ordinary cup product in cyclic cohomology [3, III.1, Thm. 12]. In [15] we used both the derived category of (co)cyclic modules and the derived category of mixed complexes to construct pairings as natural transformations of derived double functors, and we defined a comparison natural transformation between these derived functors which were isomorphisms in the cases we are interested. Independently, Rangipour developed another version of the cup product on the level of (co)cyclic modules [22] similar to [15].

In Theorem 5.12, we gave an interpretation of the cyclic cohomology computed via cyclic invariants of Hochschild cocycles as a derived functor using a relative derived category. Then in Proposition 5.13 we defined a comparison natural transformation between ordinary and relative derived functors which is an isomorphism. The primary reason we are interested in relative cyclic cohomology is the fact that (H-invariant) closed graded traces are in one-to-one correspondence (cf. [3, III.1α, Prop. 4] and [18, Lem. 2.2. and Lem. 2.3]) with cyclic- and H-invariant Hochschild cocycles which are used to implement some of the pairings we enumerated above. Thus Proposition 5.13 provides the crutial comparison natural transformation between the pairing we define in here and [15], and aforementioned pairings.

Theorem 6.4. *Let A be a H-module algebra and C be a H-module coalgebra acting on A equivariantly over H. The pairings defined in* [4, 10, 7, 18, 15, 23, 22] *are naturally isomorphic as natural transformations of isomorphic double functors.*

Proof. All of the pairings enumerate above are composed of two parts

$$\mathrm{Ext}^p(k_\bullet, X_\bullet) \otimes \mathrm{Ext}^q(Y_\bullet, k_\bullet) \xrightarrow{\eta^{p,q}} \mathrm{Ext}^{p+q}(diag_\bullet \mathrm{Hom}_k(X_\bullet, Y_\bullet), k_\bullet) \xrightarrow{\mathrm{Ext}^{p+q}(\phi, k_\bullet)} \mathrm{Ext}^{p+q}(Z_\bullet, k_\bullet)$$

(1) An *external part* which mixes a Hopf-cyclic cohomology class of the module coalgebra C and a Hopf-cyclic cohomology class of the module algebra A, as we do in Proposition 4.5, to produce an abstract cyclic cohomology class which is not necessarily a Hopf-cyclic class of a module (co)algebra

(2) An *internal part* which interprets the new class we obtained as an ordinary cyclic cohomology class of the module algebra A as we do in Theorem 6.2.

Lemma 3.9 allows us to conclude that the external parts of all such pairings agree everywhere provided that they agree on the bi-degree $(p,q) = (0,0)$. So, we consider two Hopf-cyclic classes $\alpha \in HC^0_{\rm Hopf}(C,M) = {\rm Ext}^0_\Lambda(k_\bullet, C_\bullet(C,M))$ and $\beta \in HC^0_{\rm Hopf}(A,M) = {\rm Ext}^0_\Lambda(C_\bullet(A,M), k_\bullet)$. The first class α can be represented with a k-linear morphism $\alpha'\colon A\otimes M \to k$ in ${\rm Hom}_k(A\otimes M, k)$ and the second via an element $\beta' = \sum_i c_i \otimes m_i$ in $C\otimes M$ which have invariance properties with respect to the diagonal action of H. Then the same formula which defines $HC^0(\phi_\bullet)$

$$(\alpha \smile \beta)(f) = \sum_i \alpha'(f(c_i)\otimes m_i)$$

for the specific case $f \in {\rm Hom}_k(C,A)$ given by $f(c) := c \triangleright a$ with $a \in A$, is used by Connes-Moscovici [4, VIII, Prop.1], by Gorokhovsky [10, Sect. 3, Eq. 3.11], by Crainic [7, Sect. 4.6, Eq. 20], by Khalkhali-Rangipour [18, Sect. 5], and by Nikonov-Sharygin [23, Sect. 3.3]. Thus we also observe that these pairings use the same internal part $\phi_\bullet\colon Cyc_\bullet(A) \to diag_\bullet{\rm Hom}_k(C_\bullet(C,M), C_\bullet(A,M))$ which comes from the fact that C acts on A equivariantly by [15, Prop. 2.4 and Prop. 2.7]. In [22] Rangipour splits his cup product into two pieces as we do here and in [15]. The external part of his pairing defined in [22, Sect. 2, Eq. 2.11], and the internal part defined in [22, Sect. 2, Eq. 2.13] are identical with ours. Note that even though the formulae agree, the computations are performed in different derived categories. Not all of these categories are homotopy equivalent but we have comparison natural transformations which are isomorphisms between the derived double functors evaluated on the objects we are interested (Proposition 5.13 and [15, Thm. 5.4]). $\square$

Remark 6.5. The pairing we defined in Theorem 6.2 can be easily extended to the periodic Hopf-cyclic cohomology. Note that cyclic cohomology groups computed here either via the derived functors of ${\rm Hom}_\Lambda(\,\cdot\,, k_\bullet)$ or ${\rm Hom}_\Lambda(k_\bullet,\,\cdot\,)$ are naturally graded modules over the graded algebra ${\rm Ext}^*_\Lambda(k_\bullet, k_\bullet)$ which is a polynomial algebra over one generator of degree ± 2 [1, Cor. 7], which we will denote by S. This generator implements the periodicity operator [1, Lem. 8], which really is a natural transformation of functors of the form $S\colon HC^p(\,\cdot\,) \to HC^{p\pm 2}(\,\cdot\,)$. Now using [20, Cor. 1.4] we conclude that our pairing is a morphism of S-modules, i.e. compatible with the periodicity morphism. Or we can use [23, Thm. 14] to prove the pairing defined in the derived category of mixed complexes is a morphism of S-modules, and we transport the action to the pairing defined in the derived category of Λ-modules. This means the functor HC^* in the pairing we defined above can be replaced by HP^* to obtain a periodic version.

References

[1] A. Connes. Cohomologie cyclique et foncteurs ${\rm Ext}^n$. *C. R. Acad. Sci. Paris Sér. I Math.*, 296(23):953–958, 1983.

[2] A. Connes. Noncommutative differential geometry. *Inst. Hautes Études Sci. Publ. Math.*, (62):257–360, 1985.

[3] A. Connes. *Noncommutative geometry*. Academic Press Inc., 1994 (Also available online at `http://www.alainconnes.org/docs/book94bigpdf.pdf`).

[4] A. Connes and H. Moscovici. Hopf algebras, cyclic cohomology and transverse index theorem. *Comm. Math. Phys.*, 198:199–246, 1998.

[5] A. Connes and H. Moscovici. Cyclic cohomology and Hopf algebras. *Lett. Math. Phys.*, 48(1):97–108, 1999. Moshé Flato (1937–1998).

[6] A. Connes and H. Moscovici. Cyclic cohomology and Hopf algebra symmetry. *Lett. Math. Phys.*, 52(1):1–28, 2000.

[7] M. Crainic. Cyclic cohomology of Hopf algebras. *J. Pure Appl. Algebra*, 166(1-2):29–66, 2002.

[8] J. Cuntz and D. Quillen. Cyclic homology and nonsingularity. *J. Amer. Math. Soc.*, 8(2):373–442, 1995.

[9] A. Dold and D. Puppe. Homologie nicht-additiver Funktoren. Anwendungen. *Ann. Inst. Fourier Grenoble*, 11:201–312, 1961.

[10] A. Gorokhovsky. Secondary characteristic classes and cyclic cohomology of Hopf algebras. *Topology*, 41(5):993–1016, 2002.

[11] P. M. Hajac, M. Khalkhali, B. Rangipour, and Y. Sommerhäuser. Hopf–cyclic homology and cohomology with coefficients. *C. R. Math. Acad. Sci. Paris*, 338(9):667–672, 2004.

[12] P. M. Hajac, M. Khalkhali, B. Rangipour, and Y. Sommerhäuser. Stable anti-Yetter–Drinfeld modules. *C. R. Math Acad. Sci. Paris*, 338(8):587–590, 2004.

[13] G. Hochschild. Relative homological algebra. *Trans. Amer. Math. Soc.*, 82:246–269, 1956.

[14] J. D. S. Jones and C. Kassel. Bivariant cyclic theory. *K-Theory*, 3(4):339–365, 1989.

[15] A. Kaygun. Products in Hopf-cyclic cohomology. *Homology, Homotopy and Applications*, 10(2):115–133, 2008.

[16] A. Kaygun and M. Khalkhali. Bivariant Hopf cyclic cohomology. Preprint at `arXiv:math/0606341 [math.KT]`.

[17] B. Keller. On differential graded categories. In *International Congress of Mathematicians. Vol. II*, pages 151–190. Eur. Math. Soc., Zürich, 2006.

[18] M. Khalkhali and B. Rangipour. Cup products in Hopf-cyclic cohomology. *C. R. Math. Acad. Sci. Paris*, 340(1):9–14, 2005.

[19] S. MacLane. *Homology*, volume 114 of *Die Grundlehren der mathematischen Wissenschaften*. Academic Press Inc., New York, 1963.

[20] V. Nistor. A bivariant Chern-Connes character. *Ann. of Math. (2)*, 138(3):555–590, 1993.

[21] D. Quillen. Bivariant cyclic cohomology and models for cyclic homology types. *J. Pure Appl. Algebra*, 101(1):1–33, 1995.

[22] B. Rangipour. Cup products in Hopf cyclic cohomology via cyclic modules I. Preprint at `arXiv:0710.2623v1/ [math.KT]`.

[23] G. I. Sharygin and I. Nikonov. Pairings in Hopf-cyclic cohomology of algebras and coalgebras with coefficients. Preprint at `arXiv:math/0610615v1 [math.KT]`.

[24] C. A. Weibel. *An Introduction to Homological Algebra*. Number 38 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1995.

[25] M. Wodzicki. Excision in cyclic homology and in rational algebraic K–Theory. *Ann. of Math.*, 129(3):591–639, 1989.

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